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Amplification on relativistic electron beams in combined helical and axial magnetic fields

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A free-electron laser with the guide magnetic field operating as an amplifier is analyzed. A simple dispersion relation, similar in form with or without the guide field, is derived. The study of the solutions of the dispersion relation indicates that the guide allows us to (a) enhance the spatial instability in the amplifier, and (b) significantly extend the frequency range of the instability to lower and higher frequencies. This improved operation of the amplifier with the guide field may be achieved at lower values of the pump helical magnetic field. An expression for the power gain in the amplifier as a function of its length is derived and applied in numerical examples to demonstrate the effects of the guide field.

I. INTRODUCTION

Free-electron lasers, in which the energy of a relativistic electron beam is transferred into high-frequency coherent radiation, have been studied extensively in recent years.[1] One can schematically divide free-electron-laser experiments into two groups. The first is characterized by low beam densities ($I\sim 1$ A) and high relativistic factors γ for electrons ($\gamma>20$).[2,3] In these devices collective plasma effects are usually unimportant and the single-particle theory is used to describe the interaction. The second group of experiments, for example,[4–6] uses intense electron beams ($I>1$ kA) with relatively low energies ($\gamma<10$). In such lasers the collective interaction plays the major role. An important feature of the latter group of experiments is the presence of a strong axial guide magnetic field, primarily designed to collimate the high current electron beam in the interaction region. An analysis of the effects of the presence of the guide field was recently carried out.[7–9] The single-particle theory of such lasers[8] showed that the addition of the guide field may provide a significant increase of the small signal gain due to a resonance effect between the frequency of the scattered electromagnetic wave and the natural response frequency of the steady-state electron orbits in the combined pump and guide fields. In addition, the cold fluid, fully collective theory of the laser[9] predicted an extension of the frequency range of the spatial instability.

In this paper we continue the study of the free-electron laser with a guide magnetic field and consider a conventional amplifier problem. In contrast to Ref. 9, where only the mode stability analysis was carried out, our aim will be to actually find the spatial development of the electromagnetic wave along the amplifier. We shall employ a number of physical approximations which will result in a much simpler dispersion relation than that developed in Ref. 9. The problem is thereby significantly simplified and leads to a clearer understanding of the device.

The work proceeds as follows. In Secs. II and III a system of transport equations for the amplitude of the electromagnetic wave in the amplifier is derived. Section IV deals with current sources in the transport equations by considering the momentum equation for the electrons described by the cold fluid model. A simple dispersion relation is derived in Sec. V and there its solutions are analyzed both analytically and numerically. In Sec. VI formulas for the z dependence of the amplitude of the electromagnetic wave in the amplifier are obtained. We shall simplify these formulas in several limiting cases in this section and present numerical examples. Finally, conclusions are listed and discussed in Sec. VII.

II. FIELD EQUATIONS

Consider a cold relativistic electron beam propagating along the z axis of combined helical pump and axial guide magnetic fields described by

$$\vec{\mathscr{B}}(z)=\mathscr{B}_{\perp 0}(\vec{e}_x\cos k_0 z+\vec{e}_y\sin k_0 z)+\mathscr{B}_{\parallel 0}\vec{e}_z\ , \qquad (1)$$

where $\mathscr{B}_{\perp 0}$ and $\mathscr{B}_{\parallel 0}$ are constants. The helical part of $\vec{\mathscr{B}}$ represents the field on the axis of a magnetic wiggler, commonly used in free-electron lasers and $k_0=2\pi/\lambda$, where λ is the pitch of the wiggler.

In addition to the electron beam we introduce an electromagnetic wave propagating in the same direction as the electron beam. Our aim is to solve the conventional amplifier, namely, to find the electromagnetic field at a point $z>0$ in the system if this field is known at $z=0$.

We assume that the system is infinite and homogeneous in the direction perpendicular to the guide field. Then the electromagnetic fields $\vec{E}(z,t)$, $\vec{B}(z,t)$ are described by

$$c\vec{e}_z\times\frac{\partial\vec{B}_\perp}{\partial z}=\frac{\partial\vec{E}_\perp}{\partial t}-4\pi eN\vec{V}_\perp , \quad (2)$$

$$-c\vec{e}_z\times\frac{\partial\vec{E}_\perp}{\partial z}=\frac{\partial\vec{B}_\perp}{\partial t} , \quad (3)$$

$$\frac{\partial E_z}{\partial z}=-4\pi eN , \quad (4)$$

$$B_z=0 , \quad (5)$$

where the cold fluid description of the electron beam is used and $N(z,t)$ and $\vec{V}(z,t)$ are, respectively, the electron density and the velocity field. They satisfy the continuity equation

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial z}(NV_z)=0 . \quad (6)$$

The subscript $\perp$ in (2) and (3) describes the components of the corresponding fields which are perpendicular to the z axis. Since a stationary problem is considered here, we Fourier decompose in time various time-dependent quantities and seek solutions for $\vec{E}$ and $\vec{B}$ in the form

$$\vec{E}(z,t)=\mathrm{Re}\left[\frac{mc^2}{e}\hat{E}(z)e^{-i\omega t}\right] ,$$
$$\vec{B}(z,t)=\mathrm{Re}\left[\frac{mc^2}{e}\hat{B}(z)e^{-i\omega t}\right] . \quad (7)$$

In addition, we assume that the electromagnetic field is weak enough so that it only slightly perturbes the beam and one can write

$$N(z,t)=N_0+\mathrm{Re}\left[\frac{m}{4\pi e^2}\hat{M}(z)e^{-i\omega t}\right] , \quad (8)$$

$$\vec{V}(z,t)=\vec{V}_0(z)+\mathrm{Re}[\hat{V}(z)e^{-i\omega t}] , \quad (9)$$

where $N_0=\mathrm{const}$ and $\vec{V}_0(z)$ are the density and the velocity field characterizing the beam without the presence of the electromagnetic wave and $\hat{M}$ and $\hat{V}$ are small perturbations. Equations (2) and (3) are then combined and yield on linearization

$$\frac{d^2\hat{E}_\perp(z)}{dz^2}+\frac{\omega^2}{c^2}\hat{E}_\perp(z)=i\frac{\omega}{c^4}[\omega_p^2\hat{V}_\perp(z)+\vec{V}_{0\perp}\hat{M}(z)] , \quad (10)$$

where $\omega_p^2=4\pi e^2N_0/m$. The first term on the right-hand side of the wave equation (10) represents transverse currents induced by the electromagnetic wave, while the second term describes the axial bunching of the electron density due to the ponderomotive forces of the wiggler magnetostatic field and the magnetic component of the wave. It is this bunching term which causes the free-electron-laser instability[1] and is the largest part of the source in the wave equation. The reason for the importance of the axial bunching is the strong coupling between the transverse electromagnetic wave and the axial motion of the electrons which travel with the velocities close to the phase velocity of the wave. We shall demonstrate this effect later. Nonetheless, for simplicity, already at this early stage, we neglect the first term in the source in (10) and rewrite the wave equation as

$$\frac{d^2\hat{E}_\perp(z)}{dz^2}+\frac{\omega^2}{c^2}\hat{E}_\perp(z)=i\frac{\omega}{c^4}\vec{V}_{0\perp}\hat{M}(z)=\vec{F}(z) . \quad (11)$$

The general solution of (11) is

$$\begin{aligned}\hat{E}_\perp(z)&=\hat{E}_\perp(0)\cos\frac{\omega}{c}z+\frac{\hat{E}'_\perp(0)}{\omega/c}\sin\frac{\omega}{c}z+\frac{c}{\omega}\int_0^z d\xi\,\vec{F}(\xi)\sin\frac{\omega}{c}(z-\xi)\\&=\tfrac{1}{2}\hat{E}_\perp(0)(e^{i(\omega/c)z}+e^{-i(\omega/c)z})-\frac{i\hat{E}'_\perp(0)}{2\omega/c}(e^{i(\omega/c)z}-e^{-i(\omega/c)z})\\&\quad-\frac{ic}{2\omega}\int_0^z d\xi\,\vec{F}(\xi)(e^{i(\omega/c)(z-\xi)}-e^{-i(\omega/c)(z-\xi)}) .\end{aligned} \quad (12)$$

Thus, the full wave solution for the perpendicular component of the electric field becomes

$$\frac{e\vec{E}_\perp(z,t)}{mc^2}=\frac{1}{2}\mathrm{Re}\left\{e^{i[(\omega/c)z-\omega t]}\left[\hat{E}_\perp(0)-\frac{i\hat{E}'_\perp(0)}{\omega/c}-\frac{ic}{\omega}\left[\int_0^z d\xi\,\vec{F}(\xi)e^{-i(\omega/c)\xi}-e^{-2i(\omega/c)z}\int_0^z d\xi\,\vec{F}(\xi)e^{i(\omega/c)\xi}\right]\right]+e^{-i[(\omega/c)z+\omega t]}\left[\hat{E}_\perp(0)+i\frac{\hat{E}'_\perp(0)}{\omega/c}\right]\right\}. \quad (13)$$

In order to further simplify the problem we now make the following assumptions. Firstly, consistent with the amplifier conditions, we neglect in (13) the term proportional to

$$\exp[-i(\omega z/c+\omega t)]$$

which represents a constant amplitude wave propagating in the negative z direction. Namely, we set

$$\hat{E}_\perp(0)+i\frac{\hat{E}'_\perp(0)}{\omega/c}=0\,. \quad (14)$$

Secondly, we assume that the frequency ω of the amplified electromagnetic wave is much larger than any other characteristic frequency of the system, such as the plasma frequency ω_p, the effective undulation frequency ck_0 of the wiggler, or the natural response frequency $c\mu$ of the electrons[7] (see also Sec. III). This assumption is common to many treatments of free-electron lasers, where one is usually interested in frequencies ω of the order of $2\gamma^2k_0c$ with appreciable values of the relativistic factor

$$\gamma=[1-(v/c)^2]^{-1/2}\,.$$

Exploring this disparity in frequencies we assume that

$$\hat{E}(z)=\vec{a}(z)e^{i(\omega/c)z}\,,\qquad \hat{B}(z)=\vec{b}(z)e^{i(\omega/c)z}\,, \quad (15)$$

with amplitude $\vec{a}$ and $\vec{b}$ varying on the scale much slower than that described by the exponential factors in (15), namely, in orders of magnitude

$$\frac{d\ln a}{dz},\ \frac{d\ln b}{dz}<<\frac{\omega}{c}\,. \quad (16)$$

Accordingly, one can also write $\hat{V}$ and $\hat{M}$ and therefore $\vec{F}$ in the form

$$\hat{V}(z)=c\vec{v}(z)e^{i(\omega/c)z}\,,$$
$$\hat{M}(z)=n(z)e^{i(\omega/c)z}\,, \quad (17)$$
$$\vec{F}(z)=\vec{f}(z)e^{i(\omega/c)z}=i\frac{\omega}{c}\frac{\vec{V}_{0\perp}}{c}\frac{n(z)}{c^2}e^{i(\omega/c)z}\,,$$

where $\vec{v}$, n, and $\vec{f}$ satisfy inequalities similar to (16). Then, on using (14)–(17) in Eq. (13), one gets

$$\vec{a}_\perp(z)=\vec{a}_\perp(0)-\frac{i}{2\omega/c}\left[\int_0^z d\xi\,f(\xi)-e^{-2i(\omega/c)z}\int_0^z d\xi\,f(\xi)e^{2i(\omega/c)\xi}\right]. \quad (18)$$

Differentiation of this equation yields

$$\frac{d\vec{a}_\perp}{dz}=e^{-2i(\omega/c)z}\int_0^z d\xi\,\vec{f}(\xi)e^{2i(\omega/c)\xi}\,. \quad (19)$$

Finally the equation for the z component of $\vec{a}$ is obtained from (4):

$$\left[i\frac{\omega}{c}+\frac{d}{dz}\right]a_z=-\frac{n(z)}{c^2}\,. \quad (20)$$

III. LAPLACE TRANSFORMATION OF THE FIELD EQUATIONS

In order to solve the field equations (19) and (20) we have to specify the unperturbed velocity $\vec{V}_0$ of the electrons in the beam. We will use here the results of a recent study[7] of the unperturbed orbits in free-electron lasers with the guide magnetic field. There it was demonstrated that simple helical trajectories, having the same pitch as the wiggler magnetic field and described by

$$\vec{V}_0=-w(\vec{e}_x\cos k_0z+\vec{e}_y\sin k_0z)+u\vec{e}_z\,, \quad (21)$$

are allowed in magnetic field configuration (1). In Eq. (21),

$$u=\text{const}\,,\qquad w=\frac{u\Omega_\perp/\gamma}{k_0u-\Omega_{||}/\gamma}=\text{const}\,, \quad (22)$$

where $\Omega_{\perp,||}=e\mathscr{B}_{\perp,||}/mc$. There exists the possibility of several different solutions (22) for u and w (and therefore several different orbits) for a given set of the values of $\Omega_\perp$, $\Omega_{||}$, k_0, and γ. As an example, Fig. 1 shows the axial velocity u/c versus

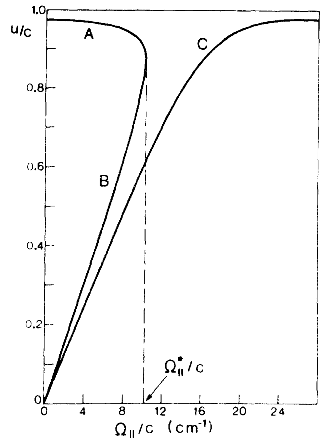

FIG. 1. Steady-state normalized axial velocity u/c vs normalized axial magnetic field $\Omega_{\|}/c$.

$\Omega_{\|}/c$ for the case $k_0=3$ cm^{-1}, $\gamma=5$, and $\Omega_{\perp}/\gamma c=0.3$ cm^{-1}. It can be seen in the figure that for $\Omega_{\|}>\Omega_{\|}^*$ only one solution (branch c) exists. But when $\Omega_{\|}<\Omega_{\|}^*$ two additional branches (A and B) are allowed. It was shown in Ref. 7 that only branches A and C are stable, against perturbations, while branch B is unstable and therefore cannot be used in applications.

We now proceed to the solution of Eqs. (19) and (20) for the fields. First, we use a more natural coordinate system in which the components of the magnetic field (1) and of the unperturbed electron velocity (21) are constants. For this purpose let

$$\vec{e}_1=-\vec{e}_x\sin k_0z+\vec{e}_y\cos k_0z\ ,$$
$$\vec{e}_2=-\vec{e}_x\cos k_0z-\vec{e}_y\sin k_0z\ , \tag{23}$$
$$\vec{e}_3=\vec{e}_z\ .$$

Then

$$\vec{\mathscr{B}}(z)=-\mathscr{B}_{\perp 0}\vec{e}_2+\mathscr{B}_{\|0}\vec{e}_3\ , \tag{24}$$

$$\vec{V}_0(z)=w\vec{e}_2+u\vec{e}_3\ , \tag{25}$$

and, on writing

$$\vec{a}=a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3\ ,$$

Eqs. (19) and (20) become

$$\frac{da_1}{dz}-k_0a_2=e^{-2i(\omega/c)z}\vec{e}_1\cdot\left[\int_0^z\vec{f}(\xi)e^{2i(\omega/c)\xi}d\xi\right]\ , \tag{26}$$

$$\frac{da_2}{dz}+k_0a_1=e^{-2i(\omega/c)z}\vec{e}_2\cdot\left[\int_0^z\vec{f}(\xi)e^{2i(\omega/c)\xi}d\xi\right]\ , \tag{27}$$

$$\left[i\frac{\omega}{c}+\frac{d}{dz}\right]a_3=-\frac{n}{c^2}\ . \tag{28}$$

This system of equations can be solved by means of a Laplace transformation. Namely, if one defines

$$a_{ik}=\int_0^\infty dz\,a_i(z)e^{-ikz},\quad n_k=\int_0^\infty dz\,n(z)e^{-ikz}\ , \tag{29}$$

where Imk is negative enough to ensure convergence, then (26)–(28) transforms into

$$ika_{1k}-k_0a_{2k}=a_1(0)+\frac{k_0}{k_0^2-[k+2(\omega/c)]^2}\frac{i\omega wn_k}{c^4}\ , \tag{30}$$

$$ika_{2k}+k_0a_{1k}=a_2(0)-\frac{[k+2(\omega/c)]}{k_0^2-[k+2(\omega/c)]^2}\frac{\omega wn_k}{c^4}\ , \tag{31}$$

$$i\left[\frac{\omega}{c}+k\right]a_{3k}=-\frac{n_k}{c^2}\ . \tag{32}$$

According to (15), $kc/\omega, k_0c/\omega<<1$, and therefore we can write the field equations in the following approximate form:

$$ika_{1k}-k_0a_{2k}=a_1(0)\ , \tag{33}$$

$$ika_{2k}+k_0a_{1k}=\frac{1}{2}\frac{wn_k}{c^3}+a_2(0)\ , \tag{34}$$

$$a_{3k}=i\frac{n_k}{\omega c}\ . \tag{35}$$

Equations (33)–(35) must be supplemented by the equation for the electron density perturbation n_k. This is obtained by taking the Laplace transformation of the linearized continuity equation (6):

$$n_k=-\omega_p^2\frac{\omega+ck}{\omega(u/c-1)+ku}v_{3k}$$
$$\simeq-\omega_p^2\frac{\omega}{\omega(u/c-1)+ku}v_{3k}\ . \tag{36}$$

It is the factor

$$\omega/[\omega(1-u/c)-ku]>>1\ ,$$

which makes the bunching in the electron beam density so important and justifies the transition from Eq. (10) to Eq. (11).

IV. MOMENTUM EQUATION

In addition to Eq. (36) for the density one has to use the momentum equation

$$\left[\frac{\partial}{\partial t}+V_z\frac{\partial}{\partial z}\right](\gamma\vec{V})=-\frac{e}{m}\left[\frac{\vec{V}}{c}\times[\vec{\mathscr{B}}(z)+\vec{B}(z,t)]+\vec{E}(z,t)\right] \tag{37}$$

in order to find v_{3k} in (36) and thus completely define the system of Eqs. (33)–(35). The components of (37) in the natural coordinate system (23) are

$$\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]V_1=V_2\left[k_0V_3-\frac{\Omega_{\|}}{\gamma}\right]-\frac{\Omega_{\perp}}{\gamma}V_3-V_1\frac{\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]\gamma}{\gamma}+\frac{e}{m\gamma}\left[\frac{V_3}{c}B_2-E_1\right], \tag{38}$$

$$\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]V_2=-V_1\left[k_0V_3-\frac{\Omega_{\|}}{\gamma}\right]-V_2\frac{\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]\gamma'}{\gamma}-\frac{e}{m\gamma}\left[\frac{V_3}{c}B_1+E_2\right], \tag{39}$$

$$\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]V_3=\frac{\Omega_{\perp}}{\gamma}V_1-V_3\frac{\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]\gamma}{\gamma}-\frac{e}{m\gamma}\left[\frac{V_1}{c}B_2-\frac{V_2}{c}B_1+E_3\right]. \tag{40}$$

Here, energy conservation yields

$$\left[\frac{\partial}{\partial t}+V_3\frac{\partial}{\partial z}\right]\gamma=-\frac{e}{mc^2}(V_1E_1+V_2E_2+V_3E_3)\,. \tag{41}$$

Linearization of (41) gives

$$\Gamma'\equiv\left[i\omega\left[\frac{u}{c}-1\right]+u\frac{d}{dz}\right]\Gamma=-wa_2-ua_3\,, \tag{42}$$

where, similar to (7) and (15), we defined

$$\gamma=\gamma_0+\mathrm{Re}[\Gamma(z)e^{-i[\omega t-(\omega/c)z]}]\,, \tag{43}$$

where γ_0 is the unperturbed relativistic factor and to orders of magnitude $d(\ln\Gamma)/dz<<\omega/c$. Linearizing Eqs. (38)–(40) and taking the Laplace transformation we get

$$i\Delta v_{1k}=av_{2k}+bv_{3k}+g\frac{\Gamma_k}{\gamma_0}+\frac{1}{\gamma_0}\left[\frac{u}{c}b_{2k}-a_{1k}\right], \tag{44}$$

$$i\Delta v_{2k}=-av_{1k}-i\Delta\frac{w}{c}\frac{\Gamma_k}{\gamma_0}-\frac{1}{\gamma_0}\left[\frac{u}{c}b_{1k}+a_{2k}\right], \tag{45}$$

$$i\Delta v_{3k}=dv_{1k}-i\Delta\frac{u}{c}\frac{\Gamma_k}{\gamma_0}+\frac{1}{\gamma_0}\left[\frac{w}{c}b_{1k}-a_{3k}\right], \tag{46}$$

where

$$\Delta=\frac{\omega}{c}\left[\frac{u}{c}-1\right]+k\frac{u}{c}$$

and

$$a=k_0u/c-\Omega_{\|}/c\gamma_0=\Omega_{\perp}u/c\gamma_0w\,, \tag{47}$$

$$b=k_0w/c-\Omega_{\perp}/c\gamma_0=\Omega_{\|}w/c\gamma_0u\,, \tag{48}$$

$$d=\Omega_{\perp}/c\gamma_0\,, \tag{49}$$

$$g=wa/c+ub/c\,. \tag{50}$$

In order to eliminate b_{1k} and b_{2k} from (44)–(46) we use Eq. (3), which reduces to

$$b_{1k}=-a_{2k}+\frac{ic}{\omega}[ika_{2k}+k_0a_{1k}-a_2(0)]\,, \tag{51}$$

$$b_{2k}=a_{1k}-\frac{ic}{\omega}[ika_{1k}-k_0a_{2k}-a_1(0)]\,. \tag{52}$$

Further simplification is possible by using (33), (34), and (36):

$$b_{1k}=-a_{2k}+\frac{iw}{2\omega}\frac{n_k}{c^2}=-a_{2k}-i\frac{\omega_p^2}{2c^3}\frac{w}{\Delta}v_{3k}\ , \tag{53}$$

$$b_{2k}=a_{1k}\ . \tag{54}$$

Finally the substitution of (53), (54), and (35) in (44)–(46) gives

$$i\Delta v_{1k}=av_{2k}+bv_{3k}+g\frac{\Gamma_k}{\gamma_0}+\left[\frac{u}{c}-1\right]\frac{a_{1k}}{\gamma_0}\ , \tag{55}$$

$$i\Delta v_{2k}=-av_{1k}-i\Delta\frac{w}{c}\frac{\Gamma_k}{\gamma_0}+\left[\frac{u}{c}-1\right]\frac{a_{2k}}{\gamma_0}+i\frac{\omega_p^2}{2c^4}\frac{uw}{\gamma_0}\frac{1}{\Delta}v_{3k}\ , \tag{56}$$

$$i\Delta v_{3k}=dv_{1k}-i\Delta\frac{u}{c}\frac{\Gamma_k}{\gamma_0}-\frac{w}{c}\frac{a_{2k}}{\gamma_0}+i\frac{\omega_p^2}{c^2\gamma_0}\left[1-\frac{w^2}{2c^2}\right]\frac{1}{\Delta}v_{3k}\ . \tag{57}$$

Equations (55)–(57) for v_{ik} $(i=1,2,3)$ are now easily solved. First, we multiply Eq. (57) by $i\Delta$ and eliminate $i\Delta u_{2k}$ and $i\Delta u_{3k}$ in the resulting equation and finally, on using Eqs. (56) and (57), we find

$$(\mu^2-\Delta^2)v_{1k}=i\Delta\left[\frac{u}{c}-1\right]\frac{a_{1k}}{\gamma_0}-S\frac{a_{2k}}{\gamma_0}+i\frac{\omega_p^2}{c^2\gamma_0}T\frac{v_{3k}}{\Delta}\ , \tag{58}$$

where

$$\mu^2=a^2-bd\ , \tag{59}$$

and

$$S=a-\left[a\frac{u}{c}-b\frac{w}{c}\right]=a-\frac{w\mu^2}{cd}\ ,$$

$$T=b+\frac{w}{2c}\left[a\frac{u}{c}-b\frac{w}{c}\right]=b+\frac{w^2\mu^2}{2c^2d}\ . \tag{60}$$

The frequency μ is the natural response frequency to external perturbations.[8] It also defines the stability of the orbits in the absence of the electromagnetic field.[7] In the limit of zero axial field, $\mu=k_0u$, namely, μ is in this case the undulation frequency of the electron beam in the wiggler. The addition of the guide field allows us to parametrically change the value of μ, and, for example, to significantly decrease it. Then, as was demonstrated recently,[8] the response of the system to perturbing electromagnetic waves becomes very strong, with a consequent increase in the gain of the amplifier. This effect of increased response at lower values of μ on stable branches A and C (see Fig. 1) is clearly seen in solution (58) for v_{1k}.

Substitution of v_{1k} from (58) into (57) results in

$$v_{3k}=\frac{iR\Delta}{\Delta^2-\eta^2}\frac{a_{2k}}{\gamma_0}-\frac{d[1-(u/c)]\Delta^2}{(\mu^2-\Delta^2)(\Delta^2-\eta^2)}\frac{a_{1k}}{\gamma_0}\ , \tag{61}$$

where

$$R=\frac{dS}{\mu^2-\Delta^2}+\frac{w}{c}\left[1-\frac{u}{c}\right]\ , \tag{62}$$

and

$$\eta^2=\frac{\omega_p^2}{c^2\gamma_0}\left[\frac{dT}{\mu^2-\Delta^2}+1-\frac{u^2}{c^2}-\frac{w^2}{2c^2}\right]\ . \tag{63}$$

V. DISPERSION RELATION

Substitution of (61) into (34) allows us to write the field equations (33) and (34) in the form

$$\epsilon_{11}a_{1k}+\epsilon_{12}a_{2k}=a_1(0)\ , \tag{64}$$

$$\epsilon_{21}a_{1k}+\epsilon_{22}a_{2k}=a_2(0)\ , \tag{65}$$

where

$$\epsilon_{11}=ik\ , \tag{66}$$

$$\epsilon_{12}=-k_0\ , \tag{67}$$

$$\epsilon_{21}=k_0-\frac{\omega_p^2\omega w}{2c^4\gamma_0}\frac{d[1-(u/c)]\Delta}{(\mu^2-\Delta^2)(\Delta^2-\eta^2)}\ , \tag{68}$$

$$\epsilon_{22}=ik+i\frac{\omega_p^2\omega w}{2c^4\gamma_0}\frac{R}{(\Delta^2-\eta^2)}\ . \tag{69}$$

Note that the resonances of $\Delta^2=\mu^2$ and $\Delta^2=\eta^2$ appear in the present theory very naturally, in contrast to the previous results,[9] where these physical effects were hidden by algebraic complexities of the reduced dielectric tensor.

Solutions of Eqs. (64) and (65) can be written

$$a_{1k}=\frac{a_1(0)\epsilon_{22}-a_2(0)\epsilon_{12}}{D}\ , \tag{70}$$

$$a_{2k}=\frac{a_2(0)\epsilon_{11}-a_1(0)\epsilon_{21}}{D}\ , \tag{71}$$

where

$$D=\epsilon_{11}\epsilon_{22}-\epsilon_{12}\epsilon_{21}$$
$$=-k^2+k_0^2-\frac{\omega_p^2\omega W}{2c^4\gamma_0}\frac{1}{(\Delta^2-\eta^2)}\left[kR+\frac{k_0 d[1-(u/c)]\Delta}{\mu^2-\Delta^2}\right]. \quad (72)$$

In order to find the z dependence of the electric field of the wave we take the inverse Laplace transformation of (70) and (71). As a preparation to this goal (which will be accomplished in Sec. VI) we shall in this section study the dispersion relation

$$D(\omega,k)=0\ , \quad (73)$$

which defines the poles of the right-hand sides of (70) and (71). We restrict ourselves to the case where the term in (72) proportional to ω_p^2 is much less than k_0^2. Then the zeros of (73) can be found by using perturbation analysis. To the lowest order, namely, for $\omega_p^2=0$, there are two roots $k=\pm k_0$. To the next order we seek solutions of the form $k=\pm k_0+x$, with $|x|<<k_0$.

First consider solutions of the form $k=-k_0+x$. Assume also that we are interested in frequencies ω which satisfy

$$\left|\omega\left[\frac{u}{c}-1\right]+k_0 u\right|<<k_0 u \quad (74)$$

or

$$\omega\simeq\frac{k_0 u}{1-u/c}=\left[1+\frac{u}{c}\right]\gamma_z^2 k_0 u=\omega_0\ , \quad (75)$$

that is, in frequencies which are close to the doubly Doppler upshifted frequency ω_0, characteristic of free electron lasers. In this case

$$\Delta=\frac{\omega}{c}\left[\frac{u}{c}-1\right]-k_0\frac{u}{c}+x\frac{u}{c}\simeq-2k_0 u/c\ ,$$

and therefore if $\mu^2<4(k_0 u/c)^2$ (this condition exists on branch A in Fig. 1, where $\mu<k_0 u$ as well as on branch C for $\Omega_{\parallel}/\gamma_0<3k_0 u$ then (73) yields

$$x_1=\frac{\omega_p^2}{16c^2\gamma_0}\frac{w\omega}{k_0^2 u^2}\left[\frac{d\{S-2[1-(u/c)]k_0(u/c)\}}{\mu^2-4k_0^2u^2/c^2}+\frac{w}{c}\left[1-\frac{u}{c}\right]\right]. \quad (76)$$

The solution is real and no instability exists for this mode. Moreover, the resonance condition $\mu^2=4k_0^2u^2$ in (76) cannot be easily achieved, so that the values of x_1 are usually so small that they hardly affect the vacuum mode at $k=-k_0$.

Now let $k=k_0+x$. Assuming again the existence of (74), we then have

$$\Delta=\beta+xu/c<<k_0 u/c\ , \quad (77)$$

where

$$\beta=\frac{\omega}{c}\left[\frac{u}{c}-1\right]+k_0\frac{u}{c} \quad (78)$$

is the mismatch frequency, characterizing the difference between ω and ω_0 [see (75)]. Thus, in this case,

$$D\simeq-2k_0 x-\frac{\omega_p^2 w\omega}{2c^4\gamma_0}\frac{k_0 R}{\Delta^2-\eta^2}\ , \quad (79)$$

where we neglected the second term in the large parentheses in Eq. (72), which is proportional to Δ. The dispersion relation then becomes

$$(\Delta-\beta)(\Delta^2-\eta^2)+\frac{\omega_p^2}{4\gamma_0}\frac{uw\omega}{c^5}R=0\ . \quad (80)$$

This dispersion relation can be easily analyzed for the case

$$\mu^2>>\Delta^2\ . \quad (81)$$

Only this case will be considered in this paper. Note that the inequality (81) still allows us to use values of μ^2 significantly lower than $(k_0 u)^2$ and thus explore the possibility of an enlarged electron response to perturbations.[8] Consistent with (81) we have

$$\eta^2=\frac{\omega_p^2}{c^2\gamma_0}\left[1-\frac{u^2}{c^2}+\frac{bd}{\mu^2}\right]=\frac{\omega_p^2}{c^2\gamma_0}\left[\frac{1}{\gamma_z^2}+\frac{bd}{\mu^2}\right] \quad (82)$$

and

$$R=\frac{ds}{\mu^2}+\frac{w}{c}\left[1-\frac{u}{c}\right]=\frac{w}{u}\left[\frac{1}{\gamma_z^2}+\frac{bd}{\mu^2}\right]. \quad (83)$$

Finally the dispersion relation (80) becomes

$$(\Delta-\beta)(\Delta^2-\eta^2)+\alpha^2\eta^2=0\ , \quad (84)$$

where

$$\alpha^2=\frac{\omega w^2}{4c^3}\ . \quad (85)$$

Note, that the form of Eq. (84) is exactly the same as the well-known and studied cubic dispersion relation for the case without the guide field. In the latter case $b=\Omega_{\parallel}w/\gamma_0 u=0$ and

$$\eta_0^2=\frac{\omega_p^2}{c^2\gamma_0}\frac{1}{\gamma_z^2}\ . \tag{86}$$

Properties of the roots of (84) in this case are well understood. For example, when $\eta << k_0 << \omega$, two roots of (84) are complex in the interval[10]

$$-\alpha^2 \lesssim \beta \lesssim \left[\frac{27\alpha^2\eta_0^2}{4}\right]^{1/3} . \tag{87}$$

At $\beta=0$ the unstable roots of (84) have a maximum imaginary part and the three roots are approximately

$$x_2=-(\alpha^2\eta_0^2)^{1/3},\quad x_{3,4}=(\alpha^2\eta_0^2)^{1/3}\left[\frac{\sqrt{3}}{2}\pm\frac{i}{2}\right] . \tag{88}$$

The presence of the guide magnetic field adds several new effects. Here the behavior of the solutions of the dispersion relation depends on the branch of the steady-state orbits (branches A and C in Fig. 1). In order to demonstrate the effect of the guide field we shall make the comparison suggested in Refs. 8 and 9, between two free-electron lasers identical except that one has an axial guide field, while the second does not. In the first laser the pump magnetic field is *reduced* so that

$$\frac{w}{u}=\frac{\Omega_\perp/\gamma_0}{k_0 u-\Omega_{\parallel}\gamma_0} \tag{89}$$

is the same in both lasers. If without the guide field $w/c=\xi/\gamma_0$ then the latter condition defines the value of the pump field for a given value of the axial field

$$\Omega_\perp=\frac{c\xi}{u}(k_0 u-\Omega_{\parallel}/\gamma_0)\ . \tag{90}$$

In our comparison the guide field affects only the parameter η^2 in (93). Thus, if $\eta^2>0$, the use of different values of the field is equivalent to the use of different beam densities. This means that for $\eta^2>0$ the general properties of the solutions of (84) for k [(87) and (88), for example] remain the same as in the case without the guide field. This effect is demonstrated in Fig. 2, where the imaginary part of the solution of (84) for k is shown as a function of ω/c for various values of the guide field in a sample case $\gamma_0=5$, $k_0=3$ cm^{-1}, $\xi=0.5$, $\omega_p^2/c^2=2$ cm^{-2} (this set of parameters is characteristic to the Naval Research Laboratory VEBA accelerator conditions). It can be seen in the figure that the parametric behavior of Imk on different branches (A or C) is different. On branch A, $bd>0$ (since $\Omega_{\parallel}/\gamma_0<k_0u$) and therefore η^2 on this branch is always larger than η_0^2 [Eq. (86)]. When μ^2 decreases, η^2 increases and so does Imk. A similar effect of an increased response was also found in the single-particle theory.[8] Consistent with (87) the upper frequency bound of the instability remains fixed in Fig. 2 and the lower frequency bound decreases with an increase of η^2. In contrast, on branch C, $bd<0$ ($\Omega_{\parallel}/\gamma_0>k_0u$) and therefore η^2 decreases as $\Omega_{\parallel}/\gamma_0$ approaches k_0u, until η^2 vanishes at $\Omega_{\parallel}/\gamma_0=1.25k_0u$. At this point the coupling between the modes in (84) disappears and so does the instability. In order to understand this effect let us again consider Eq. (57) for v_{3k} which, as we already know, defines the bunching in the electron density, responsible for the free-electron-laser instability. The first three terms in this equation are important to the discussion that follows. The parts of these terms proportional to a_{2k} describe (a) the effect of the ponderomotive force on the electrons due to the pump field, (b) the relativistic effect of the change of v_3 due to the force a_2 in the perpendicular direction, and (c) the ponderomotive force of the electromagnetic wave. It can be checked that these three factors lead to the appearance of the quantity

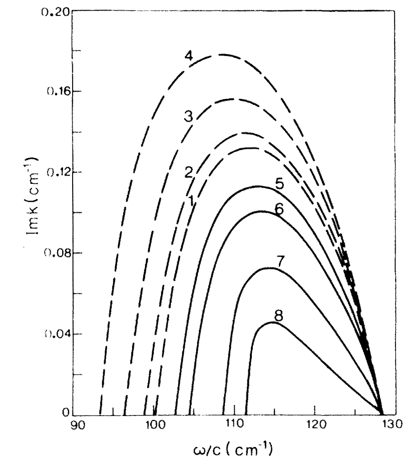

FIG. 2. Spatial growth rates Imk vs normalized frequency ω/c on branch A (dashed lines) and C (solid lines) for various values of $r=\Omega_{\parallel}/k_0u\gamma_0$: Curves 1, $r=0$; 2, $r=0.5$; 3, $r=0.8$; 4, $r=0.9$; 5, $r=2.0$; 6, $r=1.5$; 7, $r=1.3$; 8, $r=1.26$. In these calculations $\gamma_0=5$, $k_0=3$ cm^{-1}, $\xi=0.5$, and $\omega_p^2/c^2=2.0$ cm^{-2}. Note that in all the cases in the figure $\eta^2>0$.

$1/\gamma_z^2+bd/\mu^2$ in the expression (82) for η^2. On branch C the ponderomotive forces act in opposite directions. This leads to a competition and to the possibility that η^2 vanishes. By simple algebra, we find that this happens when

$$\frac{\Omega_{\parallel 0}}{\gamma_0}=k_0u(1+\xi^2) \tag{91}$$

or in our sample case when $\Omega_{\parallel 0}/\gamma_0=1.25k_0u$. This is consistent with the results in Fig. 2. At this point $\eta^2=0$ on branch C.

A new and important effect appears if one further decreases $\Omega_{\parallel}/\gamma_0$ on branch C, thus forcing η^2 to become negative. The formulas for the roots of a cubic then indicate that the region of β where (84) has complex roots is now defined by

$$\beta \gtrsim \left[\frac{27\alpha^2\eta^2}{4}\right]^{1/3}, \tag{92}$$

$$\beta \lesssim -\alpha^2 , \tag{93}$$

which is the region on the β axis complementary to the interval defined in (87). Thus the possibility of getting negative values of η^2 on branch C allows one to extend the range of the instability to both lower and higher frequencies. This effect for our sample case is demonstrated in Fig. 3, where Imk on branch C is plotted versus ω/c for several values of $\Omega_{\parallel}<\Omega_{\parallel 0}$. Note that in both regions (92) and (93) Imk approaches approximately the same value for large enough values of β. This property of the solution can be seen directly from the dispersion relation (84) which at large β is approximately

$$-\beta(\Delta^2-\eta^2)+\alpha^2\eta^2=0 .$$

The solutions

$$\Delta=\beta+ku=\pm\left[\eta^2\left[1+\frac{\alpha^2}{\beta}\right]\right]^{1/2}$$

become purely imaginary for β large enough when $\eta^2<0$ and Imk approaches $\pm(|\eta^2|)^{1/2}$.

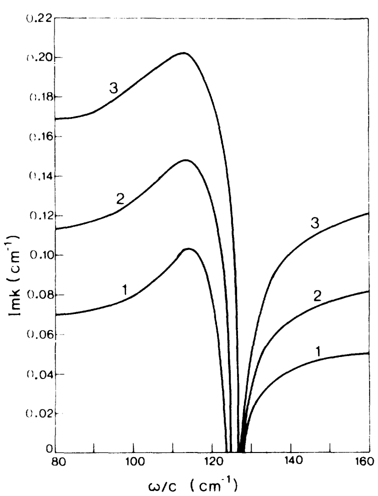

FIG. 3. Spatial growth rates Imk vs ω/c on branch C in the sample case ($\gamma_0=5$, $k_0=3$ cm^{-1}, $\xi=0.5$, $\omega_p^2/c^2=2$ cm^{-2}) in the regime $\Omega_{\parallel}<\Omega_{\parallel 0}(\eta^2<0)$. Each curve corresponds to different value of $r=\Omega_{\parallel}/k_0u\gamma_0$: Curves 1, $r=1.2$; 2, 1.15; 3, $r=1.1$.

VI. INVERSION OF LAPLACE TRANSFORMATION

The z dependence of the amplitude of the transverse electric field of the wave in the amplifier can be found on applying the inverse Laplace transformation to Eqs. (70) and (71). We write the resulting expressions in the form

$$a_1(z)=-[a_2(0)+ia_1(0)]A_1(z)+a_1(0)C_1(z) , \tag{94}$$

$$a_2(z)=i[a_2(0)+ia_1(0)]A_2(z)+a_2(0)C_2(z) , \tag{95}$$

where

$$A_1(z)=\frac{1}{2\pi}\int dk\frac{\epsilon_{12}}{D}e^{ikz} ,\qquad A_2(z)=\frac{1}{2\pi}\int dk\frac{\epsilon_{21}}{D}e^{ikz} , \tag{96}$$

and

$$C_1(z)=\frac{1}{2\pi}\int dk\frac{\epsilon_{22}+i\epsilon_{12}}{D}e^{ikz} ,\qquad C_2(z)=\frac{1}{2\pi}\int dk\frac{\epsilon_{11}-i\epsilon_{21}}{D}e^{ikz} , \tag{97}$$

and the integration in (96) and (97) is carried out in the upper half of the complex plane (Im$k>0$) and the path of the integration is taken to be above all possible poles of corresponding integrands. We will concentrate now on evaluation of the integrals A_1, A_2, C_1, and C_2 in terms of the residues of the integrands.

The poles of the integrands in (96) and (97) are defined by the roots of the dispersion relation $D=0$. It was shown in Sec. V that four such roots are of interest. One of these roots $k_1=-k_0+x_1$ is located near the point $-k_0$. The remaining three roots $k_i=k_0+x_i$ ($i=2,3,4$) are all in the neighborhood of the point k_0, so that they are well separated from k_1. Consider first the residues associated with the roots $k_{2,3,4}$. In this case in the neighborhood of these roots we write $k = k_0 + x$, with $|x| << k_0$ and thus [see (66)–(69), (82), and (85)]

$$\epsilon_{11} = ik_0 + ix \ , \tag{98}$$

$$\epsilon_{12} = -k_0 \ , \tag{99}$$

$$\epsilon_{21} \simeq k_0 \ , \tag{100}$$

$$\epsilon_{22} = ik_0 + ix + i\frac{2\alpha^2\eta^2 c}{u(\Delta^2-\eta^2)} \ . \tag{101}$$

Here, we have neglected additional term in ϵ_{21} [see (68)] as we also have done in the derivation of the dispersion relation (84). The determinant D then becomes [see Eq. (84)]:

$$D = -\frac{2k_0 c}{u(\Delta^2-\eta^2)}[(\Delta-\beta)(\Delta^2-\eta^2)+\alpha^2\eta^2] = -\frac{2k_0 c}{u(\Delta^2-\eta^2)}(\Delta-\Delta_2)(\Delta-\Delta_3)(\Delta-\Delta_4) \ , \tag{102}$$

where

$$\Delta_i = \beta + x_i u/c \quad (i = 2,3,4)$$

are the roots of the cubic (84). Thus

$$\frac{\epsilon_{12}}{D} = -\frac{\epsilon_{21}}{D} = \frac{u(\Delta^2-\eta^2)}{2c(\Delta-\Delta_2)(\Delta-\Delta_3)(\Delta-\Delta_4)} \ , \tag{103}$$

$$\frac{\epsilon_{22}+i\epsilon_{12}}{D} = -\frac{ixu(\Delta^2-\eta^2)+2i\alpha^2\eta^2 c}{2ck_0(\Delta-\Delta_2)(\Delta-\Delta_3)(\Delta-\Delta_4)} \ , \tag{104}$$

$$\frac{\epsilon_{11}-i\epsilon_{21}}{D} = -\frac{ixu(\Delta^2-\eta^2)}{2ck_0(\Delta-\Delta_2)(\Delta-\Delta_3)(\Delta-\Delta_4)} \ . \tag{105}$$

On employing the equality

$$x_i u(\Delta_i^2-\eta^2) = -\alpha^2\eta^2 c \quad (i = 2,3,4)$$

we now find the integrals associated with the modes k_2, k_3, k_4:

$$A_1^{(2,3,4)}(z) = -A_2^{(2,3,4)}(z) = -\frac{i}{2}\alpha^2\eta^2 c\left[\frac{e^{ix_2 z}}{x_2 u(\Delta_2-\Delta_3)(\Delta_2-\Delta_4)} + \frac{e^{ix_3 z}}{x_3 u(\Delta_3-\Delta_2)(\Delta_3-\Delta_4)} + \frac{e^{ix_4 z}}{x_4 u(\Delta_4-\Delta_2)(\Delta_4-\Delta_3)}\right]e^{ik_0 z} \ , \tag{106}$$

$$C_1^{(2,3,4)}(z) = -C_2^{(2,3,4)}(z) = +\frac{\alpha^2\eta^2}{2k_0}\left[\frac{e^{ix_2 z}}{(\Delta_2-\Delta_3)(\Delta_2-\Delta_4)} + \frac{e^{ix_3 z}}{(\Delta_3-\Delta_2)(\Delta_3-\Delta_4)} + \frac{e^{ix_4 z}}{(\Delta_4-\Delta_2)(\Delta_4-\Delta_3)}\right]e^{ik_0 z} \ . \tag{107}$$

Since $|x_i| << k_0$, the contribution the integrals C_1 and C_2 make in Eqs. (94) and (95) can be neglected and therefore the part of the solution for $a_1(z)$ and $a_2(z)$ associated with the modes k_2, k_3, and k_4 can be written

$$a_1^{(2,3,4)}(z) = -ia_2^{(2,3,4)}(z) = -\frac{a_1(0)-ia_2(0)}{2}\alpha^2\eta^2\left[\frac{e^{ix_2 z}}{P_2(P_2-P_3)(P_2-P_4)} + \frac{e^{ix_3 z}}{P_3(P_3-P_2)(P_3-P_4)} + \frac{e^{ix_4 z}}{P_4(P_4-P_2)(P_4-P_3)}\right]e^{ik_0 z} \ , \tag{108}$$

where $P_i = x_i u/c$.

In order to find the contribution of the remaining mode $k_1 = -k_0 + x_1$ one can use the initial conditions, rather than find the integrals (96) and (97) directly. Namely, on writing

$$a_1(z) = Q_1 e^{-ik_1 z} + a_1^{(2,3,4)}(z) \ , \qquad a_2(z) = Q_2 e^{-ik_1 z} + a_2^{(2,3,4)}(z) \ , \tag{109}$$

we find

$$Q_1=a_1(0)-a_1^{(2,3,4)}(0)=a_1(0)+\frac{a_1(0)-ia_2(0)}{2}\alpha^2\eta^2 H ,$$
$$Q_2=a_2(0)+\frac{a_2(0)+ia_1(0)}{2}\alpha^2\eta^2 H , \tag{110}$$

where

$$H=\frac{1}{P_2(P_2-P_3)(P_2-P_4)}+\frac{1}{P_3(P_3-P_2)(P_3-P_4)}+\frac{1}{P_4(P_4-P_2)(P_4-P_3)}=\frac{1}{P_2P_3P_4} . \tag{111}$$

On the other hand from (102),

$$P_2P_3P_4=(\Delta_2-\beta)(\Delta_3-\beta)(\Delta_4-\beta)=-\alpha^2\eta^2 , \tag{112}$$

and therefore

$$Q_1=iQ_2=\frac{a_1(0)+ia_2(0)}{2} . \tag{113}$$

Thus, finally, the full solutions for the amplitudes are

$$a_1(z)= -ia_2(z)=\tfrac{1}{2}[a_1(0)+ia_2(0)]e^{-i(k_0-x_1)z}$$
$$-\tfrac{1}{2}[a_1(0)-ia_2(0)]\alpha^2\eta^2\left\{\frac{e^{ix_2z}}{P_2(P_2-P_3)(P_2-P_4)}+\frac{e^{ix_2z}}{P_3(P_3-P_2)(P_3-P_4)}\right.$$
$$\left.+\frac{e^{ix_3z}}{P_4(P_4-P_2)(P_4-P_3)}\right\}e^{ik_0z} . \tag{114}$$

Assume now that initially

$$a_1(0)+ia_2(0)=0 , \tag{115}$$

namely, no electromagnetic energy is stored in the k_1 mode. Then, on using (111) and (112) we write (114) as

$$a_1(z)=-ia_2(z)=a_1(0)\left[1+\alpha^2\eta^2\left\{\frac{1-e^{ix_2z}}{P_2(P_2-P_3)(P_2-P_4)}+\frac{1-e^{ix_3z}}{P_3(P_3-P_2)(P_3-P_4)}\right.\right.$$
$$\left.\left.+\frac{1-e^{ix_4z}}{P_4(P_4-P_2)(P_4-P_3)}\right\}\right]e^{ik_0z} . \tag{116}$$

In several limiting cases this expression can be simplified and reduced to already familiar results.

(a) In the first example let $|x_iz| << 1$. In this case we expand the exponentials in (116) in powers of x_iz, by using

$$\frac{P_2^n}{(P_2-P_3)(P_2-P_4)}+\frac{P_3^n}{(P_3-P_2)(P_3-P_4)}+\frac{P_4^n}{(P_4-P_2)(P_4-P_3)}=\begin{cases}0, & n=0\\ 0, & n=1\\ 1, & n=2\end{cases} \tag{117}$$

we obtain the approximation

$$a_1(z)=-ia_2(z)=a_1(0)\left[1+i\frac{c^3\alpha^2\eta^2z^3}{6u^3}\right]e^{ik_0z} . \tag{118}$$

A similar result was obtained in Ref. 10 for the case without the guide field. In contrast to Ref. 10, however, we did not assume conditions of maximum spatial growth in the deriviation of (118).

(b) In the second example we consider the case when one of the roots of (84), say root Δ_2, is close to β (namely, $|\Delta_2-\beta| << |\beta|$) and the two remaining roots satisfy $|\Delta_3|, |\Delta_4| <<\beta$. These conditions are fulfilled when

$$\frac{\beta^2}{|\eta^2|} >> \max\left[1, \left|\frac{\alpha^2}{\beta}\right|\right] \tag{119}$$

and then

$$P_2 = x_2 u/c = \Delta_2 - \beta = -\frac{\alpha^2\eta^2}{\beta^2}, \tag{120}$$

$$P_{3,4} = x_{3,4} u/c = \Delta_{3,4} - \beta = -\beta \pm \left[\eta^2\left[1+\frac{\alpha^2}{\beta}\right]\right]^{1/2}. \tag{121}$$

Thus, in (116),

$$P_2(P_2-P_3)(P_2-P_4) \simeq -\alpha^2\eta^2, \tag{122}$$

$$P_3(P_3-P_2)(P_3-P_4) \simeq 2\beta^2\Delta_3, \tag{123}$$

$$P_4(P_4-P_2)(P_4-P_3) \simeq -2\beta^2\Delta_3. \tag{124}$$

Thus, for $|\beta z| \gtrsim 1$, we rewrite Eq. (116) in the approximate form

$$\begin{aligned} a_1(z) = -ia_2(z) &\simeq a_1(0)\left[1+i\frac{\alpha^2\eta^2}{\beta^2\Delta_3}\sin\frac{c\Delta_3 z}{u_3}e^{-ic\beta z/u}\right]e^{ik_0 z} \\ &\simeq a_1(0)\left[1+i\frac{c\alpha^2\eta^2 z}{\beta^2 u}e^{-ic\beta z/u}\right]e^{ik_0 z}. \end{aligned} \tag{125}$$

Therefore, on defining the power gain as

$$G(z) = \frac{|a_1(z)|^2 + |a_2(z)|^2}{|a_1(0)|^2 + |a_2(0)|^2} - 1 = \frac{a_1(z)a_1^*(z)}{a_1(0)a_1^*(0)} - 1 \tag{126}$$

we find from (125) that

$$G(z) \simeq 2\frac{c\alpha^2\eta^2 z}{\beta^2 u}\sin(c\beta z/u) = 2\frac{c^3\alpha^2\eta^2}{u^3}z^3\frac{\sin(c\beta z/u)}{(c\beta z/u)^2}. \tag{127}$$

The same formula for the gain was derived in the single particle, small gain theory.[8] Thus the single-particle theory corresponds to the region in parameter space defined by inequality (119), which was used in reducing Eq. (127).

We finally present a numerical example of the application of Eq. (116) in our sample case. Figure 4 shows the frequency dependence of the power gain at 25 wiggler periods for three values of $r = \Omega_{\parallel}/k_0 u\gamma_0 = 0.8$ (branch A), 1.1 and 2.0 (branch C). It follows from (91) that for $r = 0.8$ and 2, $\eta^2 > 0$, while for $r = 1.1$, $\eta^2 < 0$. It can be seen in the figure that the frequency dependence of G for positive and negative values of η^2 is completely different which reflects different type of dependence of Imk on ω (see Figs. 2 and 3). If for $\eta^2 > 0$ we see a relatively narrow frequency range for significant gain, then for $\eta^2 < 0$ this range is greatly extended. In Fig. 5 we present the z dependence of the gain in the amplifier on branch C in our sample case. The values of $r = \Omega_{\parallel}/k_0 u\gamma_0 = 1.1$ at $\omega/c = 145$ cm^{-1} (curve 1 in the figure) and $r = 2.0$ at $\omega/c = 105$, 112, and 125 cm^{-1} (curves 2a, 2b, and 2c) were again used in the calculations. The oscillations in G at short distances are due to the spatial interference of the modes in the amplifier. It is seen that only at relatively large distances does the spatial instability take over and the growth of the gain becomes exponential.

VII. CONCLUSIONS

We have the following conclusions.

(1) The free-electron-laser amplifier with a guide magnetic field was analyzed, using the cold fluid description of the electron beam.

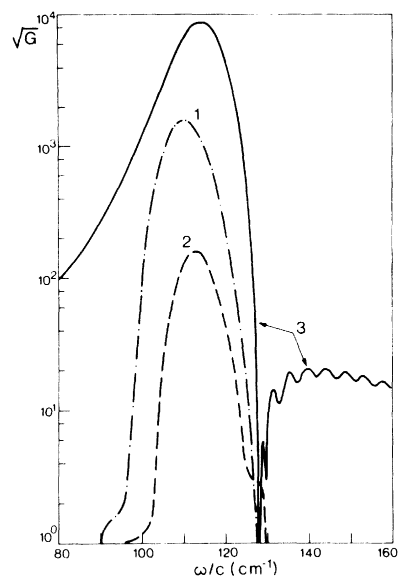

FIG. 4. Frequency dependence of the power gain in the amplifier at the distance of 25 wiggler periods in the sample case. Different curves correspond to different values $r=\Omega_{\parallel}/k_0 u\gamma_0$: Curves 1, $r=0.8$ (branch A); 2, $r=2.0$ (branch C, $\eta^2>0$); 3, $r=1.1$ (branch C, $\eta^2<0$).

(2) It was shown that in a large region of parameters space, similar to the case without the guide field, the amplified electromagnetic wave splits into four modes propagating in the direction of the electron beam. Three of the modes are coupled

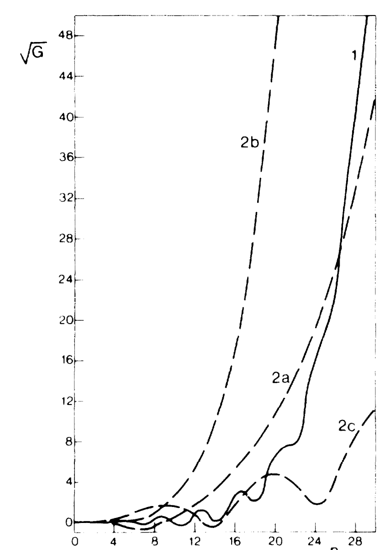

FIG. 5. Power gain in the amplifier vs the length of the interaction region measured in numbers of wiggler periods for the parameters of the sample case: Curves 1, $\Omega_{\parallel}/\gamma_0=1.1k_0u$, $\omega/c=145$ cm^{-1}; 2, $\Omega_{\parallel}/\gamma_0=2k_0u$, a, $\omega/c=105$ cm^{-1}, b, $\omega/c=112$ cm^{-1}, c, $\omega/c=125$ cm^{-1}.

and one of them may be spatially unstable. The competition between these three modes defines various regimes of operation of the amplifier.

(3) A simple dispersion relation for the coupled modes was derived and analyzed. The form of the dispersion relation is similar to the well-known cubic dispersion relation for the case without the guide field, which makes the mode stability analysis easier.

(4) The mode analysis gives the basis for the construction of the actual electromagnetic fields along the interaction region in the amplifier. Various limiting cases were considered and agreed with the results of existing theories. The power gain versus the length of the interaction region was found numerically in a sample case. The calculations demonstrated the effect of spatial interference of the modes at shorter interaction lengths and transition to exponentially growing gain at distances when the spatial instability becomes important.

(5) The main effects due to the presence of the guide field can be summarized as follows:

(i) Two types of helical orbits of the electrons can be used in the amplifier with the guide field (branches A and C in Fig. 1) for given values of γ_0, k_0, and $\Omega_{\perp}$.

(ii) On branch A the response of the electrons to electromagnetic perturbation and therefore also the spatial instability can be enhanced if the natural response frequency (see Sec. IV) of the electrons becomes small. This effect is equivalent to the increase of the density of the electron beam.

(iii) On branch C there exists an axial field $\Omega_{\parallel 0}$ [see Eq. (91)] for which the coupling between the modes disappears as well as the spatial instability. This effect is the result of the competition between the ponderomotive forces on the electron due to the pump and electromagnetic waves. For $\Omega_{\parallel}>\Omega_{\parallel 0}$ the parametric behavior of the modes is similar to that on branch A. If $\Omega_{\parallel}<\Omega_{\parallel 0}$, however, the frequency range of the instability extends significantly to both lower and higher frequencies in contrast to branch A (and branch C for $\Omega_{\parallel}>\Omega_{\parallel 0}$) where this range is relatively small and usually has an upper limit close to $\omega_0=2k_0\gamma_0^2c$.

(iv) The effects described in (i)–(iii) can be achieved for given helical orbits in the presence of the guide field at much lower values of the pump field.

ACKNOWLEDGMENTS

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